Obtain also the sums of the following terminating series and of the four series obtained from them by changing the signs of the alternate terms:

(5) $\frac{1}{n!\,1!} + \frac{1}{(n-4)!\,5!} + \frac{1}{(n-8)!\,9!} + \cdots,$

(6) $\frac{1}{(n+1)!\,0!} + \frac{1}{(n-3)!\,4!} + \frac{1}{(n-7)!\,8!} + \cdots,$

(7) $\frac{1}{(n-2)!\,3!} + \frac{1}{(n-6)!\,7!} + \frac{1}{(n-10)!\,11!} + \cdots,$

(8) $\frac{1}{(n-1)!\,2!} + \frac{1}{(n-5)!\,6!} + \frac{1}{(n-9)!\,10!} + \cdots.$

2984. Proposed by A. L. CANDY, University of Nebraska.

Find the number of numbers of n digits each that can be written with n consecutive digits, allowing all possible repetitions, such that the sum of the digits in each number is a multiple of n.

2985. Proposed by A. L. CANDY, University of Nebraska.

Find the number of combinations of n digits each that can be made with the first n consecutive digits, allowing repetitions, and such that the sum of the digits in each combination is a multiple of n.

SOLUTIONS.

2890 [1921, 184]. Proposed by B. F. FINKEL, Drury College.

Having given a triangle whose base is $2c$ and (*a*) the sum of whose other two sides is $2a$, (*b*) the difference of whose other two sides is $2a$, determine the envelope of the perpendicular bisectors of the variable sides.

I. Solution by the Proposer.

(*a*) Letting the vertices of the triangle be $(-c, 0)$, $(c, 0)$ and (α, β), we have

$$\alpha^2/a^2 + \beta^2/b^2 = 1, \tag{1}$$

where $b^2 = a^2 - c^2$.

The equation of the perpendicular bisector of the line connecting $(c, 0)$ and (α, β) may be written[1]

$$(\alpha - x)^2 + (\beta - y)^2 = (x - c)^2 + y^2. \tag{2}$$

Taking α as a variable parameter and differentiating with respect to α, we have $d\beta/d\alpha = -b^2\alpha/a^2\beta = -(\alpha - x)/(\beta - y)$, and our problem is to eliminate α and β.

The elimination may be carried out as follows:

$$\beta = b^2\alpha y/[b^2\alpha - a^2(\alpha - x)],$$

and if we write e for c/a, we have

$$\beta = \frac{(1 - e^2)\alpha y}{x - e^2\alpha} \quad \text{and} \quad \beta - y = \frac{(\alpha - x)y}{x - e^2\alpha},$$

and equations (1) and (2) become

$$a^2 - \alpha^2 = \frac{(1 - e^2)\alpha^2 y^2}{(x - e^2\alpha)^2}, \tag{3}$$

and

$$(x - ae)^2 + y^2 - (\alpha - x)^2 = \frac{(\alpha - x)^2 y^2}{(x - e^2\alpha)^2}; \tag{4}$$

whence

$$\frac{(x - ae)^2 + y^2 - (\alpha - x)^2}{a^2 - \alpha^2} = \frac{(\alpha - x)^2}{(1 - e^2)\alpha^2},$$

and therefore

$$(a^2 - e^2\alpha^2)(\alpha - x)^2 = (1 - e^2)[(x - ae)^2 + y^2]\alpha^2. \tag{5}$$

Now equation (3) is the same as

$$(a^2 - \alpha^2)(e^2\alpha - x)^2 = (1 - e^2)\alpha^2 y^2, \tag{6}$$

[1] It should be noticed that this solution, as well as the next, gives only one part of the locus, the two parts being symmetrical with respect to the y-axis—EDITORS.

and if we subtract and divide by $1 - e^2$ we get

$$(1 + e^2)a^2\alpha^2 - 2a^2x\alpha - e^2\alpha^4 + x^2\alpha^2 = (x - ae)^2\alpha^2,$$

or simplifying further and dividing also by $e\alpha - a$,

$$e\alpha^2 + a\alpha - 2ax = 0;$$

whence

$$x = \frac{(a + e\alpha)\alpha}{2a}. \tag{7}$$

Equation (6) then gives

$$y = \pm \frac{e\alpha + a - 2ae^2}{2a}\sqrt{\frac{a^2 - \alpha^2}{1 - e^2}}. \tag{8}$$

These are the parametric equations of the locus.

(*b*) The parametric equations for this part of the problem may be obtained in the same way.

II. Solution by J. K. Whittemore, Yale University.

The fixed vertices of the triangle are the foci of (*a*) an ellipse or (*b*) a hyperbola described by the variable vertex; the variable sides are the focal radii of the ellipse or hyperbola; the middle point of each focal radius describes a similar conic, the center of similitude being the corresponding focus. The problem stated is a special case of the following:

Given any curve whose polar equation is $r = f(\varphi)$ at each point P of the curve a line PQ is constructed perpendicular to OP, the radius vector of P. The envelope of PQ is required.

If the length PQ is denoted by p the rectangular coördinates of Q are

$$x = r \cos \varphi - p \sin \varphi, \qquad y = r \sin \varphi + p \cos \varphi.$$

In order that the locus of Q be the envelope of PQ it is necessary and sufficient that

$$\frac{dy}{dx} = -\cot \varphi$$

or

$$\frac{(r' - p) \sin \varphi + (r + p') \cos \varphi}{(r' - p) \cos \varphi - (r + p') \sin \varphi} = -\frac{\cos \varphi}{\sin \varphi},$$

where accents denote differentiation with respect to φ. From the last equation $p = r'$, so that the equations of the envelope in terms of the parameter φ are

$$x = r \cos \varphi - r' \sin \varphi, \qquad y = r \sin \varphi + r' \cos \varphi.$$

In the problem stated the envelope is given by substituting

$$r = f(\varphi) = \frac{1}{2}\frac{me}{1 + e \cos \varphi}, \qquad e = \frac{c}{a}, \qquad m = \pm\frac{a^2 - c^2}{c},$$

where the upper and lower signs in m correspond to the ellipse and hyperbola respectively.[1]

We may also give the intrinsic equation of the envelope in the general case. From its equations

$$dx = -(r'' + r) \sin \varphi d\varphi, \qquad dy = (r'' + r) \cos \varphi d\varphi,$$

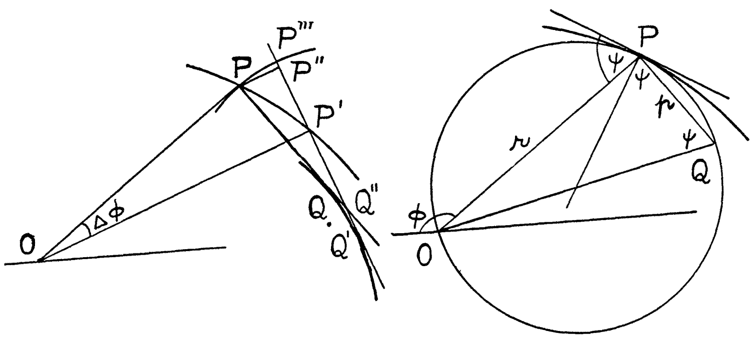
$$\rho = \frac{ds}{d\varphi} = r'' + r.$$

If the envelope is given in intrinsic form, $\rho = F(\varphi)$, the original or pedal curve is given by solving the differential equation, $r'' + r = F(\varphi)$. For example, (1) if the envelope is a point $F(\varphi) = 0$, $r = A \cos (\varphi - \alpha)$, and the pedal curve is a circle passing through the origin; (2) if the envelope is a circle $F(\varphi) = A$, $r = A + B \cos (\varphi - \alpha)$, and the pedal curve is a limaçon. Both of these results are easily verified.

Note by Otto Dunkel, Washington University—The results obtained by Professor

[1] The equations obtained in this way reduce to those given under I if we express $\cos \varphi$ and $\sin \varphi$ in terms of α and transfer the origin to the center—Editors.

Whittemore may be obtained geometrically as follows: Consider two neighboring points, P and P', on the given curve and let O be the pole, Q and Q' the points of tangency of the perpendiculars with their envelope. Then the intersection Q'' of PQ and $P'Q'$ lies on the circle through



O, P and P'. When P' approaches P as a limit the circle has for its limit a circle tangent to the given curve at P and passing through O. This gives a geometrical construction for Q. If ψ is the angle between OP and the tangent to the given curve then $\angle\, OQP = \psi$ and $\tan\psi = r/p$. But the well-known formula of polar coördnaites gives $\tan\psi = r/(dr/d\varphi)$ and hence $p = dr/d\varphi$. Consider that involute of the envelope (Q) which passes through P and let it cut $Q'P'$ in P'''; let P'' be the foot of the perpendicular from P upon $Q'P'$. Then $P'''Q' - PQ = \Delta s$ and $P'''Q' = P'''P'' + r \sin\Delta\varphi + p + \Delta p$ and hence

$$\frac{P'''P''}{\Delta\varphi} + \frac{r\sin\Delta\varphi}{\Delta\varphi} + \frac{\Delta p}{\Delta\varphi} = \frac{\Delta s}{\Delta\varphi}.$$

On taking the limit we have

$$r + \frac{dp}{d\varphi} = \rho,$$

since $P'''P''$ is equal to the length of the perpendicular from P to the tangent to the involute at P''', and hence it is an infinitesimal of the second order.

2892 [1921, 184]. Proposed by R. T. McGREGOR, Bangor, Calif.

Two parabolas have parallel axes. Prove that their common chord bisects their common tangent.

I. Solution by Marcia L. Latham, Hunter College.

Let the axes be rectangular, with the x-axis parallel to the axes of the parabolas; let P_1 and P_2, respectively, be the points of contact of the common tangent with the two parabolas, and P_3, the midpoint of P_1P_2.

Then the equations of the parabolas will be

$$y^2 + 2B_1x + 2C_1y + D_1 = 0, \tag{1}$$
$$y^2 + 2B_2x + 2C_2y + D_2 = 0. \tag{2}$$

The tangent to (1) at P_1 is

$$yy_1 + B_1(x + x_1) + C_1(y + y_1) + D_1 = 0.$$

But this passes through P_2; therefore,

$$y_1y_2 + B_1(x_1 + x_2) + C_1(y_1 + y_2) + D_1 = 0. \tag{3}$$

Again the tangent to (2) at P_2 passes through P_1; therefore,

$$y_1y_2 + B_2(x_1 + x_2) + C_2(y_1 + y_2) + D_2 = 0. \tag{4}$$

Subtracting (4) from (3), we can write

$$2(B_1 - B_2)\frac{x_1 + x_2}{2} + 2(C_1 - C_2)\frac{y_1 + y_2}{2} + (D_1 - D_2) = 0. \tag{5}$$